Pergamon
www.elsevier.com/locate/jappmathmech

PII: S0021-8928(02)00045-X

J. Appl. Maths Mechs, Vol. 66, No. 3, pp. 357–370, 2002

0021-8928/02/$—see front matter

INTERNAL RESONANCE IN AN AUTONOMOUS HAMILTONIAN SYSTEM CLOSE TO A SYSTEM WITH A CYCLIC COORDINATE†

O. V. KHOLOSTOVA

Moscow

(*Received* 11 *October* 2001)

The motion of an autonomous Hamiltonian system with two degrees of freedom, close to a system with a cyclic coordinate, is considered. It is assumed that the generating system admits of a steady rotation, the corresponding equilibrium position of the reduced system being stable in the linear approximation. It is also assumed that there is an internal resonance in the system: the ratio of the natural frequency of small oscillations of the reduced system to the frequency of variation of the cyclic coordinate is close to an integer. Non-linear oscillations of the complete system in the neighbourhood of this steady rotation are investigated. Periodic motions are constructed and their bifurcation and stability are examined. Methods of KAM theory are used to study quasi-periodic motions of the system. As an example, the problem of the motion of a nearly dynamically symmetrical heavy rigid body along an absolutely smooth horizontal plane is investigated in the case of internal resonance.

1. STATEMENT OF THE PROBLEM. TRANSFORMATION OF THE HAMILTONIAN

Consider the motion of an autonomous Hamiltonian system with two degrees of freedom. It will be assumed that the Hamiltonian of the system contains a small parameter ε ($0 < \varepsilon \ll 1$) and may be represented as

$$H = H^{(0)}(Q_2, P_1, P_2) + \varepsilon H^{(1)}(Q_1, Q_2, P_1, P_2; \varepsilon) \tag{1.1}$$

where Q_i and P_i ($i = 1, 2$) are the coordinates and momenta, respectively.

The Hamiltonian (1.1) is assumed to be a 2π-periodic function of the coordinate Q_i, but the latter occurs only in the perturbing part $\varepsilon H^{(1)}$. This coordinate is therefore cyclic for the system with the unperturbed Hamiltonian $H^{(0)}$. Suppose, moreover, that the unperturbed system admits of a steady rotation

$$Q_1 = \Omega t + Q_1^0, \;\; P_1 = P_1^0, \;\; Q_2 = P_2 = 0 \;\; (\Omega, Q_1^0, P_1^0 - \text{const}) \tag{1.2}$$

and that the equilibrium position $Q_2 = P_2 = 0$ of the reduced system is stable in the linear approximation.

We shall assume that the ratio of the natural frequency ω of small oscillations in the neighbourhood of the aforementioned equilibrium position to the frequency Ω of variation of the cyclic coordinate is close to an integer, so that there is an internal resonance in the system.

Let us consider the motion of the system with the complete Hamiltonian (1.1) in the neighbourhood of the steady rotation (1.2) of the generating system. The aim of this paper is to solve the problem of the existence, number and stability of periodic motions of the complete system with Hamiltonian (1.1), and also to investigate quasi-periodic motions of the system.

Put

$$Q_1 = q, \;\; Q_2 = \varepsilon^{1/3} q_2, \;\; P_1 = P_1^0 + \varepsilon^{2/3} P, \;\; P_2 = \varepsilon^{1/3} p_2$$

in (1.1). The Hamiltonian of the perturbed motion will have the form (omitting the additive constant)

†*Prikl. Mat. Mekh.* Vol. 66, No. 3, pp. 366–380, 2002.

$$H = \frac{1}{2}\omega(q_2^2 + p_2^2) + \Omega P + \varepsilon^{1/3}[h_3(q_2, p_2) + (l_1 q_2 + l_2 p_2)P] +$$

$$+ \varepsilon^{2/3}[h_4(q_2, p_2) + (m_1 q_2^2 + m_2 q_2 p_2 + m_3 p_2^2)P + nP^2] +$$

$$+ \varepsilon^{1/3}s(q) + \varepsilon^{2/3}[f_1(q)q_2 + f_2(q)p_2] + O(\varepsilon) \tag{1.3}$$

where $h_k(q_2, p_2)$ ($k = 3, 4$) are kth degree forms in q_2 and p_2, l_i ($i = 1, 2$), m_j ($j = 1, 2, 3$) and n are constant coefficients, and $s(q)$, $f_1(q)$ and $f_2(q)$ are 2π-periodic functions of the variable q.

We shall make a number of canonical changes of variables which simplify the structure of the Hamiltonian (1.3). First, using a univalent canonical transformation

$$q = \varphi + \ldots, \quad q_2 = q_2^* + \ldots, \quad P = I, \quad p_2 = p_2^* + \ldots$$

which is close to the identity, we transform the part of the Hamiltonian not containing the coordinate q to normal form, up to terms of the fourth order inclusive in the variables $|I|^{1/2}, q_2^*, p_2^*$. Then, changing to "polar" coordinates φ_2, I_2 by the formulae

$$q_2^* = \sqrt{2I_2}\sin\varphi_2, \quad p_2^* = \sqrt{2I_2}\cos\varphi_2$$

we write the transformed Hamiltonian as

$$H = \Omega I + \omega I_2 + \varepsilon^{2/3}(c_{20}I^2 + c_{11}II_2 + c_{02}I_2^2) +$$

$$+ \varepsilon^{1/3}s(\varphi) + \varepsilon^{2/3}\sqrt{2I_2}[f_1^*(\varphi)\sin\varphi_2 + f_2^*(\varphi)\cos\varphi_2] + O(\varepsilon) \tag{1.4}$$

where c_{ij} are constants and the functions $f_1^*(\varphi)$ and $f_2^*(\varphi)$ are 2π-periodic functions of φ.

Using the unitary canonical change of variables

$$\varphi = \varphi, \quad \varphi_2 = \varphi_2, \quad I = I^* - \varepsilon^{1/3}s(\varphi)/\Omega, \quad I_2 = I_2^*$$

we eliminate the term $\varepsilon^{1/3}s(\varphi)$ in the Hamiltonian.

Next, we simplify the terms containing the functions $f_1^*(\varphi)$ and $f_2^*(\varphi)$. Let $\omega/\Omega = N$ (where N is an integer). We represent the Fourier series of the functions $f_1^*(\varphi)$ and $f_2^*(\varphi)$ in the form

$$f_i^*(\varphi) = a_N^{(i)}\cos N\varphi + b_N^{(i)}\sin N\varphi + \sum_{n \neq N}(a_n^{(i)}\cos n\varphi + b_n^{(i)}\sin n\varphi), \quad i = 1, 2$$

A univalent canonical change of variables, 2π-periodic in φ

$$\varphi, \varphi_2, I^*, I_2^* \to \hat{\varphi}, \hat{\varphi}_2, J, J_2$$

will eliminate all terms with non-resonance harmonics in expression (1.4). There remain in the transformed Hamiltonian terms with resonance harmonics $\cos(\hat{\varphi}_2 - N\hat{\varphi})$ and $\sin(\hat{\varphi}_2 - N\hat{\varphi})$. Let us write this Hamiltonian as

$$H = \Omega J + \omega J_2 + \varepsilon^{2/3}(c_{20}J^2 + c_{11}JJ_2 + c_{02}J_2^2) + \varepsilon^{2/3}\varkappa\sqrt{J_2}\cos(\hat{\varphi}_2 - N\hat{\varphi} - \gamma) + O(\varepsilon)$$

$$\varkappa = \sqrt{2}\varkappa_1, \quad \varkappa_1 = \sqrt{A_N^2 + B_N^2}, \quad \gamma = \operatorname{arctg}\frac{B_N}{A_N}, \quad A_N = \frac{b_N^{(1)} + a_N^{(2)}}{2}, \quad B_N = \frac{a_N^{(1)} - b_N^{(2)}}{2}$$

Now make the change of variables

$$\hat{\varphi}, \hat{\varphi}_2, J, J_2 \to \varphi^*, \varphi_2^*, J^*, J_2^*$$

defined by

$$\hat{\varphi} = \varphi^*, \quad \hat{\varphi}_2 = \varphi_2^* + N\varphi^* + \gamma, \quad J = J^* - NJ_2^*, \quad J_2 = J_2^*$$

We then obtain

$$H = \Omega J^* + (\omega - \Omega N)J_2^* + \varepsilon^{2/3}(c_{20}^* J^{*2} + c_{11}^* J^* J_2^* + c_{02}^* J_2^{*2}) + \varepsilon^{2/3}\varkappa\sqrt{J_2^*}\cos\varphi_2^* + O(\varepsilon) \tag{1.5}$$

$$c_{20}^* = c_{20}, \quad c_{11}^* = c_{11} - 2Nc_{20}, \quad c_{02}^* = c_{02} - c_{11}N + c_{20}N^2$$

We now put $\omega/\Omega = N + \varepsilon^{2/3}\delta$ and rewrite the Hamiltonian (1.5) in the form

$$H = \Omega J^* + \varepsilon^{2/3}[c_{20}^* J^{*2} + (\delta\Omega + c_{11}^* J^*)J_2^* + \varkappa\sqrt{J_2^*}\cos\varphi_2^* + c_{02}^* J_2^{*2}] + O(\varepsilon)$$

Assuming that $c_{02}^* \neq 0$, we make one more change of variables

$$\varphi^*, \varphi_2^*, J^*, J_2^* \to \theta, \theta_2, \rho, \rho_2$$

by the formulae

$$\varphi^* = \theta, \; \varphi_2^* = \theta_2 + \frac{\pi}{2}(1 - \sigma), \; J^* = \varkappa_* \rho, \; J_2^* = \varkappa_* \rho_2; \; \sigma = \operatorname{sign} c_{02}^*, \; \varkappa_* = \left(\frac{\varkappa}{c_{02}^*}\right)^{2/3}$$

The transformed Hamiltonian is

$$H = \Omega\rho + \varepsilon^{2/3}\{\hat{\alpha}\rho^2 + \hat{\beta}[(a\delta + b\rho)\rho_2 + \rho_2^2 + \sqrt{\rho_2}\cos\theta_2]\} + O(\varepsilon) \tag{1.6}$$

$$\hat{\alpha} = c_{20}^* \varkappa_*, \quad \hat{\beta} = (c_{02}^* \varkappa^2)^{1/3}, \quad a = \Omega/\hat{\beta}, \quad b = c_{11}^*/c_{02}^*$$

where the term $O(\varepsilon)$ is 2π-periodic in θ, θ_2.

2. PERIODIC MOTIONS OF THE SYSTEM

2.1. *Isoenergetic reduction.* Let us consider the motions of the system with Hamiltonian (1.6) at an isoenergetic level. It follows from the equality $H = \Omega c = \text{const}$ that

$$\rho = c - \frac{\varepsilon^{2/3}}{\Omega}\{\hat{\alpha}c^2 + \hat{\beta}[(a\delta + bc)\rho_2 + \rho_2^2 + \sqrt{\rho_2}\cos\theta_2]\} + O(\varepsilon) \tag{2.1}$$

The variation of the variables θ_2 and ρ_2 is described by the Hamilton equations (Whittaker's equations); the Hamiltonian has the form

$$K = H' + O(\varepsilon^{1/3}) \tag{2.2}$$

$$H' = -\mu\rho_2 + \rho_2^2 + \sqrt{\rho_2}\cos\theta_2, \quad \mu = -(a\delta + bc) = \text{const} \tag{2.3}$$

where the independent variable is $\tau = (\varepsilon^{2/3}\hat{\beta}/\Omega)\theta$. The term $O(\varepsilon^{1/3})$ in (2.2) is periodic in τ with period $T \sim \varepsilon^{2/3}$.

Note that the resonance relation $\omega/\Omega \simeq N$ between the frequencies of the initial autonomous Hamiltonian system, which has two degrees of freedom, implies the existence in the reduced "non-autonomous" Hamiltonian system, which has one degree of freedom, of a resonance in the forced oscillations. The function H' is a model Hamiltonian of such a system [1].

Now, when resonance is present in forced oscillations of ordinary systems with one degree of freedom, the parameter μ in the model Hamiltonian defines the frequency mismatch. In the reduced system under consideration here, however, the parameter μ, besides the value of the resonance mismatch, also depends on the energy constant c of the system with two degrees of freedom.

2.2. *Motions of the model system.* The motions of the system with the model Hamiltonian (2.3) have been investigated before (see [1] and the references cited therein). Here we shall only present information necessary for what follows.

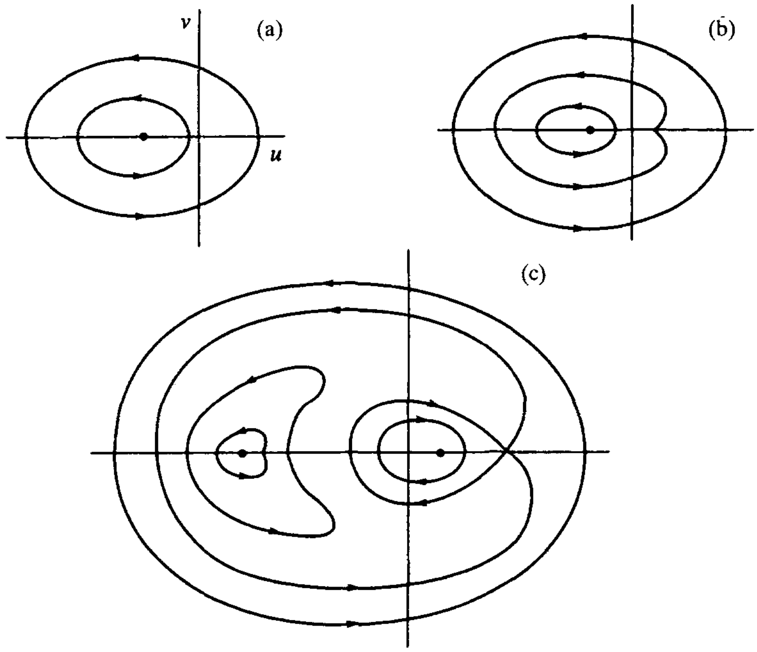

Fig. 1

Figures 1(a–c) are phase portraits of the model system in the plane of the variables $u = \sqrt{2\rho_2}\cos\theta_2$, $v = \sqrt{2\rho_2}\sin\theta_2$, for the respective cases $\mu < 3/2$, $\mu = 3/2$, $\mu > 3/2$. When $\mu < 3/2$, the system has one stable equilibrium position (Fig. 1a)

$$\rho_2^{(0)} = \frac{|\mu|}{3}\,\mathrm{ch}\frac{\varphi}{3} + \frac{\mu}{3}, \quad \theta_2^{(0)} = \pi \quad \left(\mathrm{ch}\,\varphi = \frac{27 - 4\mu^3}{4|\mu|^3}\right)$$

but when $\mu > 3/2$ it has three (Fig. 1c)

$$\rho_2^{(1)} = -\frac{\mu}{3}\cos\frac{\varphi}{3} + \frac{\mu}{3}, \quad \theta_2^{(1)} = 0; \quad \rho_2^{(2)} = -\frac{\mu}{3}\cos\left(\frac{\varphi}{3} + \frac{4\pi}{3}\right) + \frac{\mu}{3}, \quad \theta_2^{(2)} = 0$$

$$\rho_2^{(3)} = -\frac{\mu}{3}\cos\left(\frac{\varphi}{3} + \frac{2\pi}{3}\right) + \frac{\mu}{3}, \quad \theta_2^{(3)} = \pi \quad \left(\cos\varphi = \frac{4\mu^3 - 27}{4\mu^3}\right)$$

two of which, corresponding to the highest ($\rho_2^{(3)}$) and lowest ($\rho_2^{(1)}$) values of ρ_2, are stable, while one, corresponding to the middle value ($\rho_2^{(2)}$), is unstable.

When $\mu = 3/2$ (Fig. 1b) we have two equilibrium positions: a stable one $\rho_2 = 1$, $\theta_2 = \pi$ and an unstable one $\rho_2 = 1/4$, $\theta_2 = 0$.

Let h denote the energy constant of the model system ($H' = h = \text{const}$). In the plane of the parameters μ and h the aforementioned equilibrium positions are reached at points of the curves (Fig. 2)

$$h = h_i(\mu) = -\mu\rho_2^{(i)} + \rho_2^{(i)2} + \sqrt{\rho_2^{(i)}}\cos\theta_2^{(i)}, \quad i = 0, 1, 2, 3 \tag{2.4}$$

For the energy level $h = h_2(\mu)$ there are also two asymptotic motions of the system (the separatrices in Fig. 1c).

Curves (2.4), together with the straight line $\mu = 3/2$, divide the (μ, h) plane into subdomains in which the system has different types of motion.

In the G_1 and G_2 domains (Fig. 2) the system performs oscillations near the stable equilibrium positions $\rho_2 = \rho_2^{(0)}$, $\theta_2 = \pi$ (Fig. 1a) and $\rho_2 = \rho_2^{(3)}$, $\theta_2 = \pi$ (Fig. 1c), respectively. In G_3, for every point (μ, h) the system has two different motions: oscillation near the stable equilibrium position $\rho_2 = \rho_2^{(1)}$, $\theta_2 = 0$, and rotation, represented in Fig. 1c by the closed curve around the separatrix. To the energy level $h = h_1(\mu)$ there corresponds, besides the equilibrium position $\rho_2 = \rho_2^{(1)}$, $\theta_2 = 0$, a rotation-trajectory

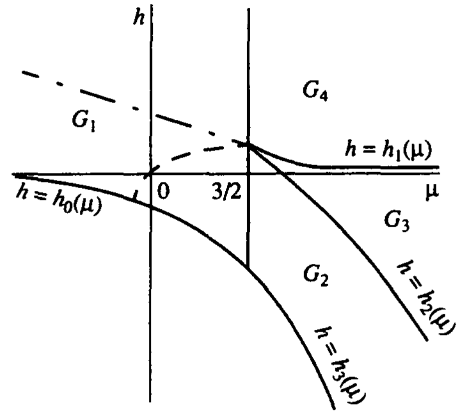

Fig. 2

(not shown in Fig. 1c). For each point of the domain G_4 there is one rotation of the system (the closed curves in Fig. 1(c) surrounding the aforementioned rotation-trajectory).

The oscillation and rotation frequencies of the model system depend on the roots of the polynomial $\varphi(\rho_2) = \rho_2 - (\rho_2^2 - \mu\rho_2 - h)^2$ [1]. In the domains G_1, G_2 and G_4 the polynomial $\varphi(\rho_2)$ has two positive real roots and a pair of complex-conjugate roots. Let a_1 and a_2 ($a_1 < a_2$) denote the real roots, and a_3 and a_4 the complex ones. The motions of the system (oscillations or rotations) in the above domains are periodic, with frequency

$$\omega_1 = \pi\sqrt{m'm''}/(2K(k_1)) \tag{2.5}$$

where

$$m' = \sqrt{a_{31}a_{42}}, \quad m'' = \sqrt{a_{41}a_{32}}, \quad a_{ij} = a_i - a_j, \quad k_1^2 = \frac{1}{2}\left(1 - \frac{m'^2 + m''^2}{2m'm''}\right)$$

and $K(k_1)$ is the complete elliptic integral of the first kind.

In G_2 the polynomial $\varphi(\rho)$ has four positive real roots $\rho_2 = a_i$ ($i = 1, 2, 3, 4$), $0 < a_1 < a_2 < a_3 < a_4$. The oscillation and rotation frequencies corresponding to each point of G_2 are the same, equalling

$$\omega_2 = \pi\sqrt{a_{42}a_{31}}/8(K(k_2)), \quad k_2 = \sqrt{(a_{43}a_{21})/(a_{42}a_{31})} \tag{2.6}$$

2.3. *Periodic solutions of the system and their stability*. According to Poincaré's theory of periodic motion [2], from each equilibrium position $\theta_2 = \theta_{2*}$, $\rho_2 = \rho_{2*}$ of the model system (except for the unstable compound singular point $\rho_2 = 1/4$, $\theta_2 = 0$ at $\mu = 3/2$) there issues a unique solution of the reduced system, analytic in $\varepsilon^{1/3}$, T-periodic in τ and therefore 2π-periodic in θ, of the form

$$\theta_2 = \tilde{\theta}_2(\theta) = \theta_{2*} + O(\varepsilon^{1/3}), \quad \rho_2 = \tilde{\rho}_2(\theta) = \rho_{2*} + O(\varepsilon^{1/3}) \tag{2.7}$$

For these solutions, it follows from (2.1) that

$$\rho = \tilde{\rho}(\theta) = c - \frac{\varepsilon^{2/3}}{\Omega}\{\hat{\alpha}c^2 + \hat{\beta}[-\mu\rho_{2*} + \rho_{2*}^2 + \sqrt{\rho_{2*}}\cos\theta_{2*}]\} + O(\varepsilon) \tag{2.8}$$

where the term $O(\varepsilon)$ is 2π-periodic in θ.

Under these conditions, the variation of θ as a function of time is described by the equation $d\theta/dt = \partial H/\partial\rho$ (where the Hamiltonian H is that defined in (1.6)), with expressions (2.7) and (2.8) substituted into the right-hand side. We have

$$d\theta/dt = \Omega_0 + \varepsilon F(\theta; \varepsilon^{1/3}), \quad \Omega_0 = \Omega + \varepsilon^{2/3}(2\hat{\alpha}c^2 + \hat{\beta}b\rho_{2*}) \tag{2.9}$$

where the function $F(\theta; \varepsilon^{1/3})$ is 2π-periodic in θ.

The solution of Eq. (2.9) may be expressed as

$$\theta = \tilde{\theta}(t) = \Omega^* t + \theta_0 + O(\varepsilon), \quad \Omega^* = \Omega_0 + O(\varepsilon) = \text{const} \tag{2.10}$$

where the term $O(\varepsilon)$ is periodic in t with period $2\pi/\Omega^*$.

Substituting expression (2.10) into equalities (2.7) and (2.8), we obtain a time-periodic solution $\tilde{\theta}_2(\tilde{\theta}(t))$, $\tilde{\rho}_2(\tilde{\theta}(t))$, $\tilde{\rho}(\tilde{\theta}(t))$ of the system with two degrees of freedom, with period

$$\tilde{T} = \frac{2\pi}{\Omega^*} = \frac{2\pi}{\Omega}\left[1 - \frac{\varepsilon^{2/3}}{\Omega}(2\hat{\alpha}c + \hat{\beta}b\rho_{2*})\right] + O(\varepsilon) \tag{2.11}$$

The number of such periodic solutions may be one or three, depending on the value of the parameter μ of the model system. In the latter case, the periods of the solutions with respect to t are identical only in the principal part; terms of order $\varepsilon^{2/3}$ and higher in (2.11) depend on the equilibrium value ρ_{2*} and therefore differ for different solutions.

Corresponding to the equilibrium position of the model system which is unstable for $\mu > 3/2$ there is an unstable periodic solution of the form (2.7), (2.8), since in that case the characteristic equation of the linearized system of equations of motion has a positive real root. Corresponding to the stable equilibrium positions of the model system are orbitally stable periodic solutions (2.7), (2.8) (with the possible exception of a set of parameter values of zero measure), which follows from the fact, to be proved in Section 3.2, that the approximate Hamiltonian (3.3) is non-degenerate and from the results of KAM theory [3].

In the initial variables, corresponding to the solutions (2.7), (2.8), which are 2π-periodic in θ, we have motions of the system with Hamiltonian (1.1) which are 2π-periodic in θ_1, namely

$$Q_2 = \varepsilon^{1/3}\sqrt{2\varkappa_*\rho_{2*}}\sin\chi + O(\varepsilon^{2/3}), \quad P_2 = \varepsilon^{1/3}\sqrt{2\varkappa_*\rho_{2*}}\cos\chi + O(\varepsilon^{2/3})$$

$$P_1 = P_1^0 + \varepsilon^{2/3}\varkappa_*(c - N\rho_{2*}) + O(\varepsilon); \quad \chi = N\theta_1 + \theta_{2*} + \pi/2(1-\sigma) + \gamma \tag{2.12}$$

where

$$\theta_1 = \bar{\Omega}t + \theta_0 + O(\varepsilon^{1/3}), \quad \bar{\Omega} = \Omega + O(\varepsilon^{1/3}) = \text{const}$$

the term $O(\varepsilon^{1/3})$ being periodic in time with period $\bar{T} = 2\pi/\bar{\Omega}$. This is also the period in t of the quantities Q_2, P_2 and P_1 in equalities (2.12).

Consequently, depending on the parameter μ of the model system, the initial system admits of either one $\bar{T}$-periodic motion of type (2.12) that is an orbitally stable motion, or three motions, two of which are orbitally stable and one unstable.

In the case of three motions, their periods differ by quantities of the order of $\varepsilon^{1/3}$. The functions $Q_2(t)$ and $P_2(t)$ for different motions have different amplitudes (depending on ρ_{2*}); the phases of the motions corresponding to $\rho_{2*} = \rho_2^{(1)}$ and $\rho_{2*} = \rho_2^{(2)}$ are identical in the principal part ($\sim\varepsilon^{1/3}$), while the phase of the third motion (for $\rho_{2*} = \rho_2^{(3)}$) is shifted by π relative to the first two. The values of the momentum P_1 for all three motions are identical in the principal part and differ in terms of the order of $\varepsilon^{2/3}$.

3. QUASI-PERIODIC MOTIONS OF THE SYSTEM

3.1. *The existence of quasi-periodic motions*. If the term $O(\varepsilon)$ is dropped from (1.6), we obtain an approximate Hamiltonian. The coordinate θ in the approximate system is cyclic, and the corresponding momentum is constant: $\rho = c_* = \text{const}$.

Let us write the approximate Hamiltonian as

$$\tilde{H} = \Omega c_* + \varepsilon^{2/3}(\hat{\alpha}c_*^2 + \hat{\beta}H') \tag{3.1}$$

where H' is the model Hamiltonian defined in (2.3), in which $\mu = -(a\delta + bc_*)$. Note that in this expression for μ the quantity c_* may be replaced by the energy constant c of the complete system (as in (2.3)), since the two constants c and c_* differ by a quantity of the order of $\varepsilon^{2/3}$ (see (2.1)) and the Hamiltonian H' in (3.1) is contained in a term $\sim\varepsilon^{2/3}$.

For the approximate Hamiltonian (3.1) we introduce action-angle variables I, W and I_2, w_2 in each of the domains of oscillations or rotations of the model system with Hamiltonian H'. We put

$$I = \rho = c_*, \quad I_2 = \frac{1}{2\pi}\oint \rho_2 d\theta_2 = I_2(h,\mu) \tag{3.2}$$

when the integration is carried out along a closed trajectory of the model system corresponding to oscillation or rotation, and we use the energy integral $H' = h = \text{const}$.

If the function $I_2 = I_2(h, \mu)$ is inverted, we obtain the Hamiltonian H' expressed in terms of action-angle variables

$$h = h(I_2,\mu) = h(I_2,\mu(\delta,c_*)) = h(I_2,\mu(\delta,I)) = \tilde{h}(I,I_2)$$

The approximate Hamiltonian $\tilde{H}$ may therefore be expressed as

$$\tilde{H}(I,I_2) = H^{(0)}(I) + \varepsilon^{2/3} H^{(1)}(I,I_2) \tag{3.3}$$

$$H^{(0)}(I) = \Omega I, \quad H^{(1)}(I,I_2) = \hat{\alpha} I^2 + \hat{\beta}\tilde{h}(I,I_2)$$

We shall also consider the system with the complete Hamiltonian written in terms of the variables I, w, I_2, w_2.

$$H = \tilde{H}(I,I_2) + \varepsilon H^{(2)}(I,I_2,w,w_2;\varepsilon^{1/3}) \tag{3.4}$$

The Hamiltonian (3.4) is analytic in all its arguments, except at the singular points and separatrices of the model system.

The following conditions hold for the approximate Hamiltonian (3.30)

$$1)\frac{\partial H^{(0)}}{\partial I} \neq 0, \quad 2)\frac{\partial H^{(1)}}{\partial I_2} \neq 0, \quad 3)\frac{\partial^2 H^{(1)}}{\partial I_2^2} \neq 0$$

Condition *1* may be verified directly; condition *2* follows from formulae (2.5) and (2.6). Condition *3* is equivalent to the non-degeneracy condition $\partial^2 h/\partial I_2^2 \neq 0$ for the model Hamiltonian. This will be verified in Section 3.2; it will be shown that the model Hamiltonian satisfies the non-degeneracy condition for all admissible values of the parameters μ and h, except for the set of points (μ, h) on the curve represented in Fig. 2 by the dashed curve.

Based on the results of KAM-theory [3], for all initial conditions, the variables I and I_2 in the complete system with Hamiltonian (3.4) always remain in the vicinity of their initial values. For most initial conditions, the motions of the complete system will be quasi-periodic with frequencies $\tilde{\Omega} = \Omega + O(\varepsilon^{2/3})$ and $\tilde{\Omega}_2 = \varepsilon^{2/3}\hat{\beta}\partial\tilde{h}/\partial I_2 + O(\varepsilon)$, where $d\tilde{h}/\partial I_2$ is the oscillation or rotation frequency of the model system, as defined by (2.5) or (2.6); the following estimates are then valid:

$$|I(t) - I(0)| \sim O(\varepsilon^{1/3}), \quad |I_2(t) - I_2(0)| \sim O(\varepsilon^{1/3})$$

(except for values of μ and h outside a small neighbourhood of the dashed curve in Fig. 2). The addition to the aforementioned majority of initial conditions is of the order of $e^{-d/\varepsilon^{2/3}}$, where $d = \text{const} > 0$.

3.2. *Verification of the non-degeneracy condition for the model Hamiltonian.* We shall now verify that the model Hamiltonian $h(I_2, \mu)$ satisfies the non-degeneracy condition in the domains $G_1, \ldots, G_4$ of the parameter plane (μ, h) (Fig. 2) described in Section 2.2.

We first consider the rotation domains G_3 and G_4, as well as the part of the oscillation domain G_1 to whose points the trajectories in Fig. 1a surrounding the origin correspond.

Using the second relation of (3.2) and the energy integral of the model system, we write

$$\frac{d^2h}{dI_2^2} = \frac{\omega^3}{2\pi}\oint \frac{\partial^2 H'/\partial\rho_2^2}{(\partial H'/\partial\rho_2)^3} d\theta_2 = \frac{\omega^3}{2\pi}\oint \frac{2 - \frac{1}{4}\cos\theta_2 \rho_2^{-3/2}}{(-\mu + 2\rho_2 + \frac{1}{2}\cos\theta_2\rho_2^{-1/2})^3} d\theta_2 \tag{3.5}$$

where $\omega = dh/dI_2$ is the frequency of the oscillation or rotation in question.

Since the angle θ_2 increases monotonically on the relevant trajectories, it follows that $\partial H'/\partial\rho_2 = d\theta_2/d\tau > 0$ and the denominator of the fraction in (3.5) is positive. As for the numerator, it is positive on trajectories for which $\rho_2 > 1/4$ at all angles θ_2. Under those conditions $d^2h/dI_2^2 > 0$, and the non-degeneracy condition holds.

Among the trajectories corresponding to oscillations at $\mu = 3/2$ there is one passing through the point $\rho_2 = 1/4$, $\theta_2 = 0$. On this trajectory $1/4 \leqslant \rho_2 \leqslant s_1$, where s_1 is the unique root of the equation

$$\rho_2^2 - \sqrt{\rho_2} - \mu\rho_2 = (9 - 4\mu)/16$$

in the domain under consideration. On all trajectories surrounding this one we have $\rho_2 > 1/4$, and the non-degeneracy condition holds. The points of the domain G_1 corresponding to these trajectories are those above the straight line $h = \mu/4 + 9/16$, which is represented in Fig. 2 by a dash-dot line.

Now let $\mu \geqslant 3/2$. When $\mu = 3/2$ (Fig. 1b) were have $1/4 < \rho_2 \leqslant 9/4$ on the separatrix, but when $\mu > 3/2$ (Fig. 1c) the following relation holds on the outer loop of the separatrix

$$\rho_2^{(2)} < \rho_2 \leqslant s_2, \quad s_2 = \rho_2^{(2)} + \mu + \sqrt{2(\mu^2 - \sqrt{\rho_2^{(2)}})}$$

and moreover the estimate $\rho_2^{(2)} > \mu/6 > 1/4$ holds [1].

Thus, when $\mu \geqslant 3/2$, we have $\rho_2 > 1/4$ on all rotation-trajectories of the model system, and therefore $d^2h/dI_2^2 > 0$ and the non-degeneracy condition holds.

We note, moreover, that the non-degeneracy condition is also satisfied in the oscillation domain in the neighbourhood of the stable equilibrium $\rho_2 = \rho_2^{(1)}$, $\theta_2 = \theta_2^{(1)}$, to which the points of the already considered domain G_3 correspond: the oscillation and rotation frequencies ω_2 in G_3 are the same (see formula (2.6)), and since $d^2h/dI_2^2 = d\omega_2/dI_2 = \omega_2 d\omega_2/dh$ the non-degeneracy condition reduces to the inequality $d\omega_2/dh \neq 0$, which is the same for oscillations and rotations in the domain G_3.

We will now verify the non-degeneracy condition for points of the domain G_2 and the part of the domain G_1 below the straight line $h = -\mu/4 + 9/16$.

For points of a small neighbourhood of the curve $h = h_2(\mu)$ in G_2, corresponding to oscillations near the separatrices, we set $h = h_2(\mu) - \Delta$ $(0 < \Delta \leqslant 1)$, and it then follows from (2.5) that $\omega_2 \approx -a/\ln\Delta$, $a = \text{const} > 0$. Hence if follows that

$$d\omega_1 / dh = -d\omega_1 / d\Delta \approx -a/(\Delta \ln^2 \Delta) < 0$$

and, consequently, the non-degeneracy condition holds in the neighbourhood of the separatrices of the oscillation domain G_2.

For points near the curves $h = h_0(\mu)$ and $h = h_3(\mu)$ of the domains G_1 and G_2, corresponding to oscillations in the neighbourhood of the stable equilibria $\rho_2 = \rho_2^{(0)}$, $\theta_2 = \theta_2^{(0)}$ and $\rho_2 = \rho_2^{(3)}$, $\theta_2 = \theta_2^{(3)}$, we have

$$h = h_k(\mu) + B_1 I_2 + B_2 I_2^2 + O(I_2^3), \quad k = 0, 3 \tag{3.6}$$

$$B_1 = (\mu - 2\rho_2^{(k)})^2 + 2\sqrt{\rho_2^{(k)}}, \quad B_2 = [(\mu - 2\rho_2^{(k)})^4 - \tfrac{1}{2}\sqrt{\rho_2^{(k)}}(\mu - 2\rho_2^{(k)})^2 - \rho_2^{(k)}]/B_1$$

If $B_2 \neq 0$, the Hamiltonian (3.6) satisfies the non-degeneracy condition. This condition is violated if

$$(\mu - 2\rho_2^{(k)})^2 = \frac{\sqrt{17}+1}{4}\sqrt{\rho_2^{(k)}}, \quad k = 0, 3 \tag{3.7}$$

In the domain G_2 we have $B_2 < 0$ for all $\mu = 3/2$. In the domain G_1, Eq. (3.7) is satisfied when $\mu = \mu_* = -0.188959\ldots$. If $\mu_* < \mu < 3/2$, then $B_2 < 0$; when $\mu < \mu_*$, we have $B_2 > 0$.

For points of the domains G_1 and G_2 outside small neighbourhoods of the above-mentioned boundaries, the validity of non-degeneracy condition for the model Hamiltonian has been verified on a computer. The computations showed that $d^2h/dI_2^2 < 0$ for all points of G_2 (as also for points of that domain near the curves $h = h_2(\mu)$, $h = h_3(\mu)$, as verified analytically above). At the same time, a curve exists in the domain G_1 (shown in Fig. 2 by the dashed curve) on which the non-degeneracy condition fails to hold. This curve begins on the curve $h = h_0(\mu)$ at $\mu = \mu_*$, passes through the origin, and ends at the point $(3/2, 3/16)$ on the boundary line $\mu = 3/2$ (from which the boundary curves $h = h_1(\mu)$

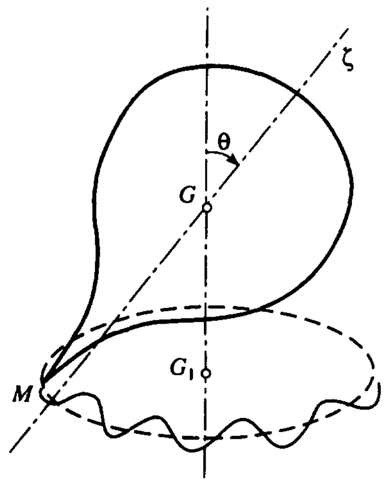

Fig. 3

and $h = h_2(\mu)$ of the domains G_2, G_3 and G_4 issue). For points of the domain G_1 to the right of the curve, $d^2h/dI_2^2 < 0$; for points to its left, $d^2h/dI_2^2 > 0$.

4. EXAMPLE: A BODY WITH A SHARP POINT ON A SMOOTH SURFACE

4.1. *Formulation of the problem.* Consider the motion of a heavy rigid body on a stationary, absolutely smooth, horizontal plane, assuming that there is a sharp point on its surface (that is, the radius of curvature of the surface vanishes there), moving along this plane. The sharp point is close to one of the principal central axes of inertia of the body. The body's moments of inertia relative to the other two principal central axes of inertia are assumed to be close to one another.

Let $Oxyz$ be a fixed system of coordinates whose origin lies on the supporting plane, with the Oz axis directed vertically upward. Let $G\xi\eta\zeta$ be a system of coordinates attached to the body, with its origin at the centre of mass G of the body and its axes directed along the principal central axes of inertia of the body. Let x, y, z denote the coordinates of the point G; the body's orientation will be given by the Euler angles ψ, θ and φ.

Suppose the sharp point M of the body is near the $G\zeta$ axis (Fig. 3); let ξ, η and ζ denote the coordinates of M in the coordinates system of $G\xi\eta\zeta$ (ξ and η are assumed to be small) and l is the distance GM; then $\xi^2 + \eta^2 + \zeta^2 = l^2$. Then distance from the centre of mass to the supporting plane is

$$GG_1 = f(\theta, \varphi) = -\xi \sin\theta \sin\varphi - \eta \sin\theta \cos\varphi - \zeta \cos\theta$$

The kinetic and potential energies of the body may be written as [4]

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}(A\cos^2\varphi + B\sin^2\varphi + mf_\theta^2)\dot{\theta}^2 + \frac{1}{2}(C + mf_\varphi^2)\dot{\varphi}^2 +$$
$$+ \frac{1}{2}[(A\sin^2\varphi + B\cos^2\varphi)\sin^2\theta + C\cos^2\theta]\dot{\psi}^2 + mf_\theta f_\varphi \dot{\theta}\dot{\varphi} +$$
$$+ (A - B)\sin\theta \sin\varphi \cos\varphi \dot{\theta}\dot{\psi} + C\cos\theta\dot{\varphi}\dot{\psi}, \quad \Pi = mgf(\theta, \varphi) \tag{4.1}$$

where m is the mass of the body, and A, B and C its principal central moments of inertia ($A \simeq B$).

It follows from formulae (4.1) that the coordinates x, y and ψ are cyclic. Consequently, the quantities $\dot{x}, \dot{y}$ and the momentum p_ψ are constant, and the problem reduces to investigating a system with two degrees of freedom with generalized coordinates θ and φ. We may assume without loss of generality that $\dot{x} = \dot{y} = 0$.

We change to canonical variables, introducing generalized momenta

$$p_\psi = \partial T / \partial\dot{\psi}, \quad p_\theta = \partial T / \partial\dot{\theta}, \quad p_\varphi = \partial T / \partial\dot{\varphi}$$

Using the factor $ml\sqrt{lg}$, we make the momenta non-dimensional, also introducing dimensionless time $\tau = \sqrt{g/l}t$ and governing parameters ε, β, γ, a and b by the formulae

$$\varepsilon = \frac{A-B}{B} \; (0<\varepsilon\ll 1), \quad \beta = \frac{B}{ml^2}, \quad \gamma = \frac{C}{ml^2}, \quad \xi = \varepsilon la, \quad \eta = \varepsilon lb$$

Using certain relations established in [4], one can derive the following expression for the Hamiltonian of the reduced system (retaining the previous notation for the momenta and treating p_ψ as a parameter)

$$H = H^{(0)} + \varepsilon H^{(1)} + O(\varepsilon^2) \tag{4.2}$$

$$H^{(0)} = \frac{1}{2}\left[\frac{p_\theta^2}{\beta+\sin^2\theta} + \frac{p_\varphi^2}{\gamma} + \frac{(p_\varphi\cos\theta - p_\psi)^2}{\beta\sin^2\theta}\right] + \cos\theta$$

$$H^{(1)} = -\frac{\beta}{2}\left[\cos\varphi\frac{p_\theta}{\beta+\sin^2\theta} - \sin\varphi\frac{p_\varphi\cos\theta - p_\psi}{\beta\sin\theta}\right]^2 + \tag{4.3}$$

$$+\frac{1}{\beta\gamma}(-a\cos\varphi + b\sin\varphi)[(\beta\sin^2\theta + \gamma\cos^2\theta)p_\varphi - \gamma\cos\theta p_\psi]\frac{p_\theta}{\beta+\sin^2\theta} -$$

$$-\frac{1}{2}(a\sin\varphi + b\cos\varphi)\sin 2\theta\frac{p_\theta^2}{(\beta+\sin^2\theta)^2} - (a\sin\varphi + b\cos\varphi)\sin\theta$$

The system corresponding to Hamiltonian (4.2) is close to a system with a cyclic coordinate, since the coordinate φ occurs only in the perturbed part.

4.2. *The case* ε = 0. *Regular precession of the body.* The case ε = 0 corresponds to the motion of a dynamically symmetrical body ($A = B$) with a sharp point on its axis of dynamical symmetry ($\xi = \eta = 0$). For a system with approximate Hamiltonian (4.3) we have p_φ = const. Let us assume further that the values of the constant quantities p_φ and p_ψ are identical, setting $p_\varphi = p_\psi = \alpha$. The Hamiltonian $H^{(0)}$ may be written in the form

$$H^{(0)} = \frac{1}{2}\left[\frac{p_\theta^2}{\beta+\sin^2\theta} + \frac{\alpha^2}{\beta}\mathrm{tg}^2\frac{\theta}{2}\right] + \cos\theta \tag{4.4}$$

The system with Hamiltonian (4.4) has particular solutions (equilibrium positions) of the form $p_\theta = 0$, $\theta = \theta_0$ = const, where $\theta_0 = 0$ or θ_0 is a root of the equation $\cos^4(\theta/2) = \alpha^2/(4\beta)$. The first of these corresponds to motion in which the body is rotating about a vertically positioned axis of symmetry at a constant angular velocity (a "sleeping" top). This motion is stable with respect to the variables θ and p_θ, provided that $\alpha^2 > 4\beta$; this condition may be rewritten in the form

$$C^2 r_0^2 > 4Bmgl \tag{4.5}$$

where r_0 is the angular velocity of rotation of the body about the vertical. Inequality (4.5) is an analogue of the well-known Maiyevskii–Chetayev condition for the stability of a "sleeping" Lagrange top with a fixed point.

The second of the above-mentioned equilibrium positions of the approximate system exists and is stable provided that $\alpha^2 < 4\beta$, that is, provided that the reverse inequality to (4.5) holds. Corresponding to this equilibrium position is the regular precession of the dynamically symmetrical body; the constant angular velocities of precession ψ′ and proper rotation φ′ are given by the formulae

$$\psi' = \frac{\alpha}{2\beta\cos^2(\theta_0/2)}, \quad \varphi' = \Omega = \frac{\alpha}{\gamma} - \frac{\alpha\cos\theta_0}{2\beta\cos^2(\theta_0/2)} \tag{4.6}$$

where the prime denotes differentiation with respect to τ.

In the case of regular precession, the centre of mass G of the body remains fixed, while the point M describes a circle of radius $l\sin\theta_0$ in the plane of its motion (the dashed curve in Fig. 3).

Note that, since the angle θ at which the axis of symmetry of the body is inclined to the vertical cannot exceed $\pi/2$, it follows that $1/4 < \cos^4(\theta_0/2) < 1$, and therefore regular precession exists provided that $\beta < \alpha^2 < 4\beta$. We may assume here that $\alpha > 0$, since if α is replaced by $-\alpha$ a change only occurs in the sense of rotation of the body (the signs of the angular velocities in (4.6)), not in the magnitudes of these angular velocities.

4.3. *Transformation of the perturbed Hamiltonian*. Let us take the solution

$$p_\theta = 0, \quad \theta = \theta_0 = 2\arccos[\alpha^2/(4\beta)]^{1/4}, \quad p_\varphi = \alpha$$

of the approximate system as the unperturbed solution and consider the motions of the complete system with Hamiltonian (4.2) in the neighbourhood of this motion. Setting

$$\theta = \theta_0 + \varepsilon^{1/3}x_2, \quad p_\theta = \varepsilon^{1/3}y_2, \quad \varphi = q, \quad p_\varphi = \alpha + \varepsilon^{2/3}P$$

in (4.2) and making the substitution

$$x_2 = a_* q_2, \quad y_2 = p_2/a_*, \quad a_* = (\omega_0^2\beta_*)^{-1/4}, \quad \omega_0^2 = 4\sin^2(\theta_0/2), \quad \beta_* = \beta + \sin^2\theta_0$$

we reduce the part of the Hamiltonian that is quadratic in x_2 and y_2 to normal form. The transformed Hamiltonian is

$$H = \frac{1}{2}\omega(q_2^2 + p_2^2) + \Omega P + \varepsilon^{1/3}(a_1 q_2^3 + a_2 q_2 p_2^2 + a_3 q_2 P) + \varepsilon^{2/3}(b_1 q_2^4 + b_2 q_2^2 p_2^2 +$$

$$+ b_3 q_2^2 P + b_4 P^2) - \varepsilon^{1/3}(a\sin\theta_0 \sin q + b\sin\theta_0\cos q + {1/2}\sin^2\theta_0\sin^2 q) -$$

$$- \varepsilon^{2/3}[(f_1\sin q + f_2\cos q)q_2 + (f_3\sin q + f_4\cos q)p_2 + d_1\sin^2 q q_2 + d_4 \sin q\cos q p_2] + O(\varepsilon)$$

where

$$\omega = \frac{\omega_0}{\sqrt{\beta_*}}, \quad a_1 = a_*^3\,\mathrm{tg}(\theta_0/2), \quad a_2 = -\frac{\sqrt{\omega_0}\sin 2\theta_0}{2\beta_*^{7/4}}, \quad a_3 = \frac{a_*\sin\theta_0}{\alpha}$$

$$b_1 = \frac{a_*^4(6+\omega_0^2)(2+\omega_0^2)}{96\cos^2(\theta_0/2)}, \quad b_2 = -\frac{\beta\cos 2\theta_0 - \sin^2\theta_0(3 - 2\sin^2\theta_0)}{2\beta_*^3}$$

$$b_3 = \frac{a_*^2(2+\omega_0^2)}{4\alpha}, \quad b_4 = \frac{1}{2}\left[\frac{1}{\gamma} + \frac{\mathrm{ctg}^2\theta_0}{\beta}\right], \quad d_1 = \alpha a_3, \quad d_2 = \alpha\omega_0^2 a_1, \quad f_1 = a_* a\cos\theta_0$$

$$f_2 = a_* b\cos\theta_0, \quad f_3 = -bf_*, \quad f_4 = af_*, \quad f_* = \frac{\alpha[\beta\sin^2\theta_0 - \gamma\cos\theta_0(1-\cos\theta_0)]}{\alpha_*\beta\gamma\beta_*^2}$$

We now use the Deprit–Hori method to reduce the part of the Hamiltonian not containing the coordinate q to normal form up to and including fourth-order terms, and then, by the canonical change of variables

$$a,\ p,\ q_2,\ p_2 \to \hat\varphi,\ J,\ \hat q_2,\ \hat p_2$$

(as in Section 2), eliminate terms $\sim\varepsilon^{1/3}$ containing the coordinate q. After transforming to "polar" coordinates $\hat\varphi_2, J_2$ by the formulae

$$\hat q_2 = \sqrt{2J_2}\sin\hat\varphi_2, \quad \hat p_2 = \sqrt{2J_2}\cos\hat\varphi_2$$

and eliminating the term in the Hamiltonian with the non-resonance harmonic $\sin\hat\varphi$, we obtain (retaining the previous notation for the variables)

$$H = \Omega J + \omega J_2 + \varepsilon^{2/3}(c_{20}J^2 + c_{11}JJ_2 + c_{02}J_2^2) + \varepsilon^{2/3}[\varkappa_1^+ \cos(\hat\varphi_2 - \hat\varphi - \delta_1^+) +$$
$$+ \varkappa_1^- \cos(\hat\varphi_2 + \hat\varphi - \delta_1^-) + \varkappa_2^+ \cos(\hat\varphi_2 - 2\hat\varphi - \delta_2^+) + \varkappa_2^- \cos(\hat\varphi_2 + 2\hat\varphi - \delta_2^-)] + O(\varepsilon) \quad (4.7)$$
$$\varkappa_1^\pm = \sqrt{\frac{1}{2}[(f_1 \pm \hat f_4)^2 + (f_2 \mp \hat f_3)^2]}, \quad \varkappa_2^\pm = |\varkappa_{2*}^\pm|, \quad \varkappa_{2*}^\pm = \frac{\sqrt{2}}{4}(\hat d_2 \pm d_1)$$
$$\delta_1^\pm = \operatorname{arctg}\frac{f_2 \mp \hat f_3}{\hat f_4 \pm f_1}, \quad \delta_2^\pm = \pm\frac{\pi}{2}\operatorname{sign}\varkappa_{2*}^\pm$$
$$\hat f_3 = f_3 + \frac{a_3 b}{\omega}\sin\theta_0, \quad \hat f_4 = f_4 - \frac{a_3 a}{\omega}\sin\theta_0, \quad \hat d_2 = d_2 - \frac{a_3}{\omega}\sin^2\theta_0$$

where, as shown by calculations

$$c_{20} = \frac{1}{2\gamma} + \frac{1 - 4\sin^2(\theta_0/2)}{8\beta\sin^2(\theta_0/2)}, \quad c_{11} = -\frac{\cos^2(\theta_0/2)(\beta + 4\sin^4(\theta_0/2))}{2\alpha\beta_*^{3/2}\sin(\theta_0/2)}$$
$$c_{02} = -\frac{4 + 3\operatorname{ctg}^2(\theta_0/2)}{16\beta_*} + \frac{\sin^2\theta_0 + 3\cos\theta_0}{4\beta_*^2} - \frac{\beta\cos^2\theta_0}{4\beta_*^3}$$

4.4. *Conditions for the existence of resonances.* Since $\omega > 0$, while Ω may take values of any sign, it follows that when $\Omega > 0$ the system admits of resonances $\omega \approx \Omega$ and $\omega \approx 2\Omega$, and when $\Omega < 0$, resonances $\omega \approx -\Omega$ and $\omega \approx -2\Omega$. The resonance relations $\omega \approx N\Omega$ ($N = 1, 2, -1, -1$) occur when

$$\gamma = \gamma_N(\alpha, \beta) = \frac{2\alpha\beta\sqrt{\beta_*}\cos^2(\theta_0/2)}{(2/N)\beta\sin\theta_0\cos(\theta_0/2) + \alpha\sqrt{\beta_*}\cos\theta_0} \quad (4.8)$$

The quantities γ_1 and γ_2 in (4.8) are always positive for $\alpha > 0$, so that the resonances $\omega \approx 2\Omega$ and $\omega \approx \Omega$ hold at all points (α, β) of the domain $\beta < \alpha^2 < 4\beta$ or, what is the same, the domain $\sqrt{\beta} < \alpha < 2\sqrt{\beta}$ (Fig. 4) of existence of regular precession. The quantities γ_{-1} and γ_{-2}, however, change sign on passing through the curves $\alpha = \alpha_{-1}(\beta)$ and $\alpha = \alpha_{-2}(\beta)$ defined implicitly in the (α, β) plane by the relations

$$(2/N)\sin\theta_0\cos(\theta_0/2) = -\alpha\cos\theta_0\sqrt{\beta + \sin^2\theta_0} \quad (\cos^4(\theta_0/2) = \alpha^2/(4\beta)) \quad (4.9)$$

The curves $\alpha = \alpha_{-1}(\beta)$ and $\alpha_{-2}(\beta)$ are shown in Fig. 4 by the upper and lower dashed curves, respectively. In subdomain 1 in Fig. 4 we have $\gamma_{-2} < 0$, and in subdomains 1 and 2, $\gamma_{-1} < 0$. Hence the only resonances occurring in subdomain 1 are $\omega = \Omega$, $\omega = 2\Omega$, in subdomain 2 – the resonances $\omega = \Omega$, $\omega = \pm 2\Omega$, while in subdomain 3 all four resonances $\omega = \pm 2\Omega$, $\omega = \pm\Omega$ are possible (but these resonances pertain to different bodies, since they correspond to different values of the parameter γ).

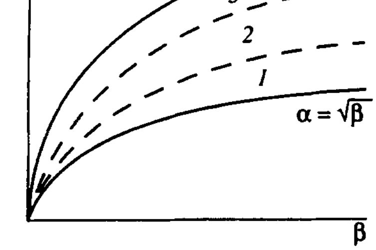

Fig. 4

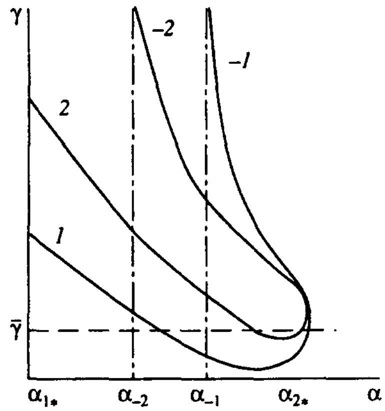

Fig. 5

For a physical interpretation of the conditions for resonances to exist, we will describe these results in a slightly different way. Fix the parameter $\beta = \bar\beta$ and consider the functions $\gamma = \Gamma_N(\alpha) \equiv \gamma_N(\alpha, \bar\beta)$ $(N = 1, 2)$ for $\alpha_{1*} \leqslant \alpha \leqslant \alpha_{2*}$ $(\alpha_* = \sqrt{\bar\beta},\ \alpha_{2*} = 2\sqrt{\bar\beta})$ and $\gamma = \Gamma_M(\alpha) \equiv \gamma_M(\alpha, \bar\beta)$ for $\alpha_M < \alpha < \alpha_{2*}$ $(\alpha_M = \alpha_M(\bar\beta), M = -1, -2)$.

All four functions $\gamma = \Gamma_N(\alpha)$ $(N = -2, -1, 1, 2)$ have a unique common point $(2\sqrt{\bar\beta}, 2\sqrt{\bar\beta})$ on the right boundary of their domains of definition (the point where the graphs of the functions have vertical tangents). For all other values of α in the domains of definition of these functions we have $\Gamma_{-1}(\alpha) > \Gamma_{-2}(\alpha) > \Gamma_2(\alpha) > \Gamma_1(\alpha)$.

The function $\gamma = \Gamma_1(\alpha)$ has a minimum point for $0 < \bar\beta < 2/7$, and the function $\gamma = \Gamma_2(\alpha)$ has one for $0 < \bar\beta < 8$; the abscissa of the minimum point is determined by the following implicit equation

$$\alpha\bar\beta(1+\sin^2(\theta_0/2)) + 3\alpha\sin^2(\theta_0/2)\sin^2\theta_0 = N\sin\theta_0\cos(\theta_0/2)\bar\beta_*^{3/2}, \quad N = 1, 2$$

where $N = 1$ for $\gamma = \Gamma_1(\alpha)$ and $N = 2$ for $\gamma = \Gamma_2(\alpha)$. For $\bar\beta \geqslant 2/7$ and $\bar\beta \geqslant 8$ the minimum point lies outside the domains of definition of the functions $\gamma = \Gamma_1(\alpha)$ and $\gamma = \Gamma_2(\alpha)$, respectively; when $\bar\beta$ varies in these intervals, these functions increase monotonically. At the left boundary of the domain of definition we have $\Gamma_N(\alpha_{1*}) = (2/N)\sqrt{2\bar\beta(\bar\beta+1)}$ $(N = 1, 2)$; the function $\gamma = \Gamma_1(\alpha)$, considered in the interval $0 < \bar\beta < 1/7$ $(\bar\beta \geqslant 1/7)$, satisfies the inequalities $\Gamma_1(\alpha_{1*}) > \Gamma_1(\alpha_{2*})$ $(\Gamma_1(\alpha_{1*}) \leqslant \Gamma_1(\alpha_{2*}))$, while the function $\gamma = \Gamma_2(\alpha)$ in the interval $0 < \bar\beta < 1$ $(\bar\beta \geqslant 1)$ satisfies the inequalities $\Gamma_2(\alpha_{1*}) > \Gamma_2(\alpha_{2*})$ $(\Gamma_2(\alpha_{1*}) \leqslant \Gamma_2(\alpha_{2*}))$.

The functions $\gamma = \Gamma_{-1}(\alpha)$ and $\gamma = \Gamma_{-2}(\alpha)$ decrease monotonically everywhere in their domains of definition.

Figure 5 illustrates the data of the analysis of resonance cases for values of $\bar\beta$ in the interval $(0, 1/7)$ (similar analyses may be carried out for other values of $\bar\beta$). The solid curves in Fig. 5 are graphs of the functions $\gamma = \Gamma_N(\alpha)$ $(N = -2, -1, 1, 2)$. The numbers on the curves correspond to the subscript N.

Fix the parameters $\beta = \bar\beta$ and $\gamma = \bar\gamma$ defining the body. In order to determine at what angular velocities of precession and intrinsic rotation (characterized by the parameter α) resonances of the form $\omega = N\Omega$ $(N = -2, -1, 1$ or $2)$ occur, we must find the number and abscissae of the points at which the straight line $\gamma = \bar\gamma$ intersects the graphs of the functions $\gamma = \Gamma_N(\alpha)$. One of these straight lines is shown in Fig. 5 by the dashed line; it intersects each of the graphs of the functions $\gamma = \Gamma_1(\alpha)$ and $\gamma = \Gamma_2(\alpha)$ at two points. For the selected values of $\bar\beta$ and $\bar\gamma$, therefore, there are two resonances of type $\omega = \Omega$ and two resonances of type $\omega = 2\Omega$, corresponding to different values of the parameter α (different angular velocities). For other values of $\bar\gamma$ the number of resonance cases varies from zero to four.

4.5. *Periodic motions of the body.* Now suppose the parameters α, β and γ are such that one of the relations $\omega \simeq N\Omega$ $(N = -2, -1, 1$ or $2)$ holds. Then, following the algorithm described in Sections 1 and 2, we obtain the following motions, 2π-periodic in φ, that are close to regular precession: the angle θ at which the $G\zeta$ axis is inclined to the vertical is defined by the following relation (θ_{2*} and ρ_{2*} are the equilibrium values of the variables θ_2 and ρ_2 of the model system)

$$\theta = \theta(\varphi) = \theta_0 + \varepsilon^{1/3} a_* \sqrt{2\varkappa_* \rho_{2*}} \sin\left(N\varphi + \theta_{2*} + \frac{\pi}{2}(1-\sigma) + \delta\right) + O(\varepsilon^{2/3})$$

where $\delta = \delta_N^+$ or $\delta = \delta_N^-$ ($N = 1$ or 2), $\varkappa_* = (\varkappa/c_{02}^*)^{2/3}$, and $\varkappa = \varkappa_N^+$ or $\varkappa = \varkappa_N^-$; the number c_{02}^* is computed by the formula in (1.5), $\sigma = \operatorname{sign} c_{02}^*$.

The angular velocities $\psi'(\varphi)$ and $\varphi'(\varphi)$ of precession and proper rotation differ from their unperturbed constant values (4.6) by 2π-periodic corrections of order $\varepsilon^{1/3}$ and higher. If ε is sufficiently small, $\psi'(\varphi)$ is of constant sign (in particular, $\psi' > 0$ for $\alpha > 0$); hence the point M of the body describes the curve shown in Fig. 3 in its plane of motion. The points of this curve deviate from a circle of radius $l\sin\theta_0$ (the trajectory of the point M in unperturbed motion) by a quantity of the order of $\varepsilon^{1/3}$ and higher.

The dependence of the angle φ on the time τ in these motions is given by

$$\varphi(\tau) = \overline{\Omega}\tau + \varphi_0 + O(\varepsilon^{1/3}), \quad \overline{\Omega} = \Omega + O(\varepsilon^{1/3}) = \text{const}$$

in which the term $O(\varepsilon^{1/3})$ is periodic in τ with period $\overline{T} = 2\pi/\overline{\Omega}$. The quantities θ, ψ' and φ' have the same periods as functions of τ.

Depending on the size of the resonance mismatch and the energy constant (both determined by the value of the parameter μ of the model system), one or three such periodic motions of the body exist that are close to regular precession.

In the case when one such motion exists (for $\mu < 3/2$), it is stable with respect to the variables $\theta, p_\theta, p_\varphi$; if three motions exist ($\mu < 3/2$), two of them, corresponding to the least and greatest amplitude of deviation from the unperturbed value of θ_0, are stable, while the third, corresponding to the middle amplitude, is unstable.

REFERENCES

1. KHOLOSTOVA, O. V., The motion of a Hamiltonian system with one degree of freedom in the case of resonance in forced oscillations. *Izv. Ross. Akad. Nauk. MTT*, 1996, 3, 167–175.
2. MALKIN, I. G., *Some Problems in the Theory of Nonlinear Oscillations*. Gostekhizdat, Moscow, 1956.
3. ARNOL'D, V. I., KOZLOV, V. V. and NEISHTADT, A. I., *Mathematical Aspects of Classical and Celestial Mechanics (Advances in Science and Technology. Ser. Modern Problems in Mathematics. Basic Trends*, Vol. 3). VINITI, Moscow, 1985.
4. MARKEYEV, A. P., *The Dynamics of a Body in Contact with a Solid Surface*. Nauka, Moscow, 1992.

Translated by D.L.